Effect of Noise on Patterns Formed by Growing Sandpiles

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Abstract. We consider patterns generated by adding large number of sand grains at a single site in an abelian sandpile model with a periodic initial configuration, and relaxing. The patterns show proportionate growth. We study the robustness of these patterns against different types of noise, *viz.*, randomness in the point of addition, disorder in the initial periodic configuration, and disorder in the connectivity of the underlying lattice. We find that the patterns show a varying degree of robustness to addition of a small amount of noise in each case. However, introducing stochasticity in the toppling rules seems to destroy the asymptotic patterns completely, even for a weak noise. We also discuss a variational formulation of the pattern selection problem in growing abelian sandpiles.

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1. Introduction

The problem of how a large animal develops from a single cell has been a central problem in biology. A somewhat simpler, but nontrivial, problem is how a baby animal grows into an adult, increasing the total body mass by two or three orders of magnitude, while different parts of the body keep roughly same proportions during growth. We will refer to this property as *proportionate growth*. Our work has been motivated by trying to construct minimal physical models with this property.

A simple example of proportionate growth in a non-biological context is a dew drop on a windowpane. Its shape may be approximately described as a spherical frustum, where the contact angle with the glass surface is determined by the surface tension. As it takes water from the super-saturated air in the surrounding, it grows in size, and shows nearly proportionate growth. However, it is not easy to construct models showing proportionate growth in patterns with substructures. In fact, no other model of growth studied in physics literature so far, shows this property. In the well studied Eden model [1], diffusion-limited aggregation [2], invasion percolation [3], or the Kardar-Parisi-Zhang type models [4], growth occurs in the outer "active regions", whereas the inner core once formed, remains essentially frozen afterwards.

It seems reasonable that a proportionate growth would require some central regulation or a long-distance communication and coordination between different parts of the structure. For an animal growth this is certainly true. The growth is orchestrated by the turning on and off of different regulatory enzymes and chemicals, ultimately determined by the genetic program encoded in the cell's DNA. It is interesting that, such growths can be achieved in a model system with components of much lower complexity, *i.e.*, a cellular automaton model with only a few states per site. In an earlier work [5], we have studied the Abelian Sandpile Model (ASM) as the prototypical model of proportionate growth. The patterns are formed by adding large number of sand grains at a single site on a periodic initial configuration (also referred to as "background"), and letting the system relax to a stable configuration. We were able to characterize the full pattern analytically in one simple case. The effect of adding absorbing sites, or multiple centers of growth was studied in [6].

Real biological growth occurs in a fairly noisy environment. While deterministic cellular automaton models with simple toppling rules cannot be considered realistic biologically, it is still interesting to ask whether the patterns produced by growing sandpiles are robust against introduction of a small amount of noise. In the presence of noise, an analytical study of this problem is quite difficult. The techniques used in [5] to characterize the pattern exactly, no longer work, as they depend on the potential function in each patch being a quadratic function of the coordinates (see section 2). The work reported here is exploratory, and mainly numerical. However, the fact that the patterns show some degree of robustness, even in the presence of noise, strongly suggests that this is not a special property related to the exact solubility of the ASM, and a more general macroscopic description of pattern formation and pattern selection

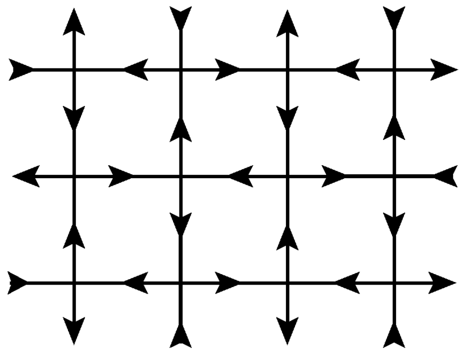

Figure 1. The F-lattice: A square lattice with directed edges whose directions are assigned periodically as shown.

in this problem, not requiring an exact solution, should be possible. We discuss how the "least action principle" for ASMs could provide a possible framework for understanding pattern formation in our problem, as the variational formulation provides a quantitative criterion for pattern selection.

We have studied the robustness of these patterns against different types of noises, namely, random fluctuations in the position of the point of addition of grains, disorder in the periodic background configuration of heights, and disorder in the connectivity of the underlying background lattice. We find that the patterns show a varying degree of robustness to addition of a small amount of noise in each case. However, introducing stochasticity in the toppling rules seems to destroy the asymptotic pattern completely, even for a weak noise.

The spatial patterns formed in sandpile models were first studied by Liu *et.al.* [7]. The asymptotic shapes of the boundaries of sandpile patterns produced by adding grains at single site on different periodic backgrounds was discussed in a later work by Dhar [8]. Borgne *et.al.* [9] obtained some bounds on the rate of growth of these boundaries, and later these bounds were improved by Fey *et.al.* [10] and Levine *et.al.* [11]. The first detailed analysis of different periodic structures found in the sandpile patterns was discussed by Ostojic [12]. An extensive collection of centrally seeded sandpile patterns on different lattices, with high resolution images, can be seen in [13]. Other special configurations in the ASMs, like the identity [9, 14, 15] or the stable state produced from special unstable states [7], also show complex structures , which share common features with the single source patterns studied here.

This paper is organized as follows. In section 2, we define the models precisely, and introduce the scaled excess density function and the scaled toppling function. These functions give a quantitative characterization of the patterns. In section 3, we discuss the robustness of the patterns against small fluctuations in the position of the point of addition. In section 4, we discuss the effect of noise in the background configuration. In section 5, we discuss the effect of disorder in the underlying lattice on which the growth occurs. This is modeled by a quenched disorder in the toppling rules. We find that, the

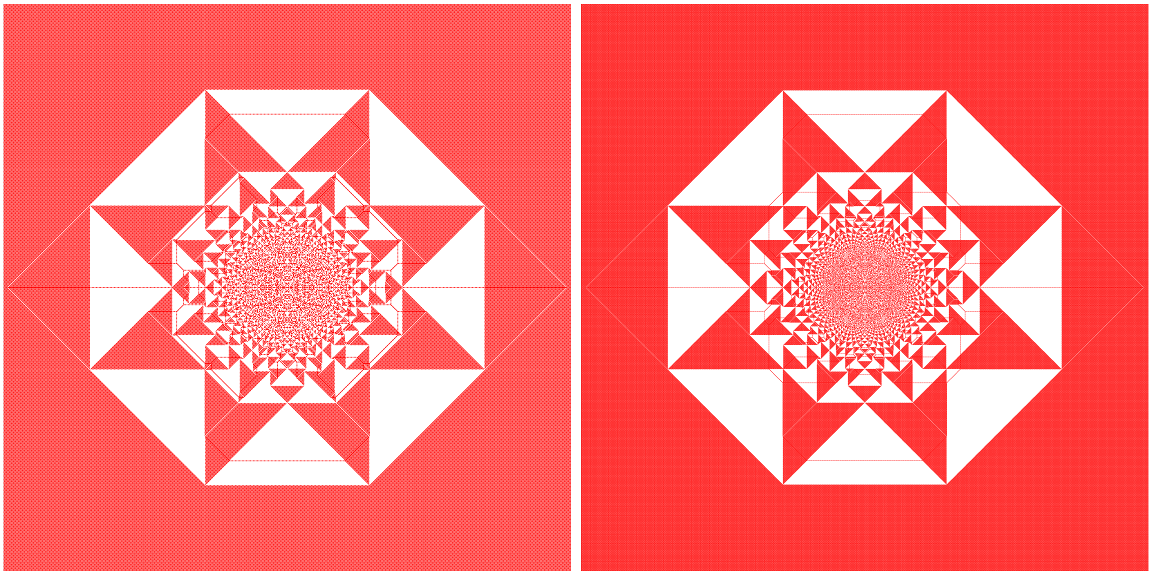

Figure 2. The patterns formed in the ASM defined on the F-lattice with checkerboard background of heights 0 and 1. These two patterns correspond to $N = 80,000$ and $N = 320,000$ grains, respectively. Color code: Red= 0, White= 1. For comparison, the size of the first pattern has been enlarged by a factor two.

patterns are quite sensitive to changes in the toppling matrix. In section 6, we discuss the effect of noise due to fluctuations in the toppling process, *i.e.*, at each toppling, there is a small probability that the grain transfer occurs in a direction not given by the toppling rule. We find that, even a very small amount of noise in the toppling rules completely wipes out the asymptotic pattern. Finally, in the concluding section 7, we discuss the 'least action principle' for the ASM, and suggest that it could provide a basic framework for understanding pattern formation in these systems.

2. Preliminaries and definitions

For our numerical studies, we have used two model systems (Here the term "model system" is used as in biology literature: the fruit fly is a model animal, and biological functions in other animals are qualitatively similar).

The first model is defined on an infinite square lattice with directed edges, such that at each site there are two edges coming in, and two going out (figure 1). This directed square lattice is called the F-lattice. At each site $\mathbf{x}$, there is a non-negative integer variable $z(\mathbf{x})$, called the number of sand grains at $\mathbf{x}$, also called the height of the sandpile at that site. Any site with height greater than 1 is said to be unstable, and it topples by transferring exactly two grains in the direction of outward arrows from that site. We start with an initial configuration in which the heights 0 and 1 form a checkerboard pattern. At each time step, we add a grain at the origin, and let the resulting configuration relax until all sites are stable. After N grains have been added, with N large, we find that the heights form an intricate and beautiful pattern, whose

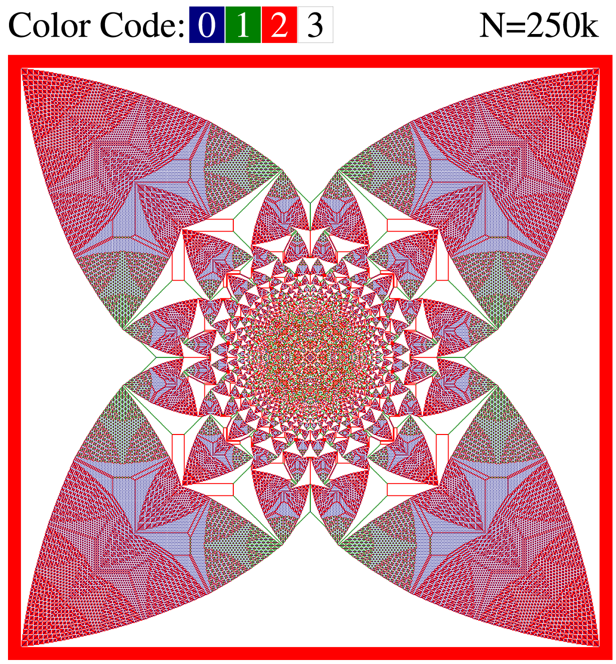

Figure 3. A pattern produced on a square lattice with a background of height 2 at every site.

size grows as N increases. The resulting patterns for $N = 80,000$ and $320,000$ are shown in figure 2. Note that, what appears to be solid red region in the figure due to low resolution, is actually a checkerboard pattern of alternate red and white squares. Details may be seen by zooming in. We see that the two scaled patterns are the same, except that the smaller patches close to the center of the pattern are resolved better in the second.

The second model system is the ASM on an undirected infinite square lattice. We define the ASM on this lattice as follows: in a stable configuration, all sites have heights less than 4. Any site where the height is greater than 3 is said to be unstable, and it relaxes by transferring 4 sand grains from that site, one to each of the four nearest neighbors. We start with an initial configuration where the height at each site is 2. The stable configuration after adding $N = 250,000$ grains at the origin is shown in figure 3.

One can consider patterns obtained when the initial configuration has a different periodic structure. For the undirected square lattice, when the initial configuration is a periodic arrangement of heights, in which each site has height less than or equal to 2, one finds a pattern in which the diameter of the pattern grows as $\sqrt{N}$ [16]. For the F-lattice, there are some backgrounds on which the growth of the pattern is faster than $\sqrt{N}$ [17]. In all the cases studied so far, if there are no infinite avalanches, the patterns show proportionate growth. Although we do not have a rigorous proof of this important property, there is good numerical evidence, and we shall assume it in the following.

A key observation is that, for large N, the patterns in figure 2 or figure 3 may be

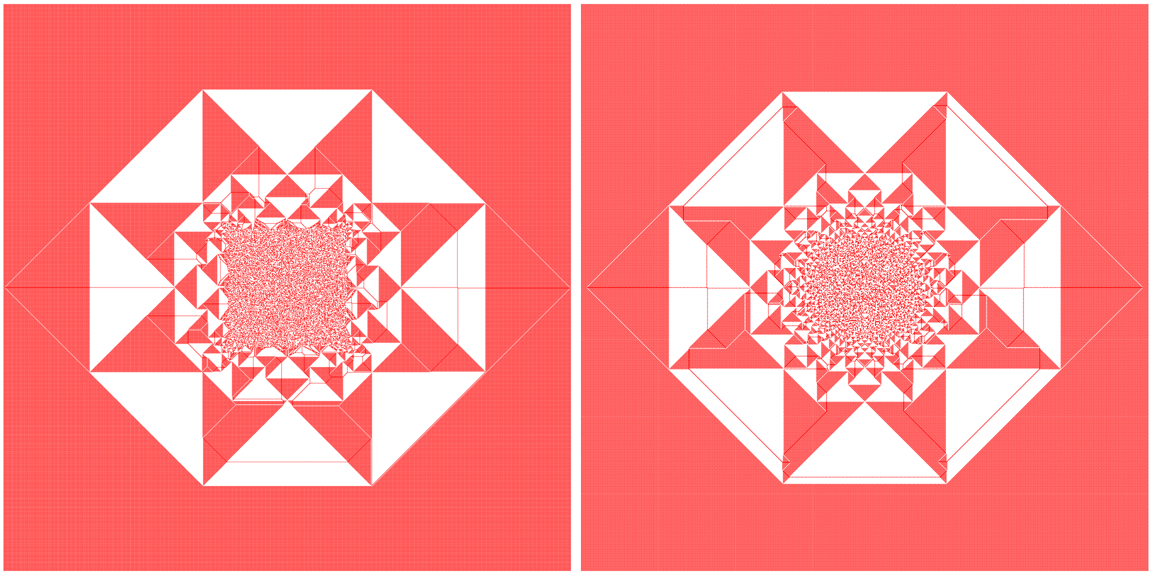

Figure 4. The patterns produced on the F-lattice by adding sand grains at sites randomly chosen from a square region of area equal to 9% and 1%, respectively, of that of the corresponding final patterns. The initial configuration is with checkerboard distribution of heights 0, and 1. Color code: Red=0, white=1.

seen as a union of disjoint patches, each of which occupies a non-zero fraction of the area of the full pattern, and the arrangement of heights inside a single patch is exactly periodic. We denote the diameter of the pattern by Λ, which may be defined as the width of the smallest square enclosing the pattern. We define reduced coordinates $\xi = x/\Lambda$ and $\eta = y/\Lambda$. The local excess density of grains $\Delta\rho(\xi, \eta)$ is defined as the difference in the density of grains in the final and initial patterns, in a small neighborhood of the point (ξ, η) in the reduced coordinates. We specify the asymptotic pattern in the limit of large N, by specifying the function $\Delta\rho(\xi, \eta)$. From the fact that inside each patch, there is a periodic pattern of integer heights, it follows that the excess density $\Delta\rho(\xi, \eta)$ is a rational constant for each patch.

It is useful to define a function $\Phi(\xi, \eta)$ as the scaled number of topplings at the site with reduced coordinates (ξ, η). Let $T_N(x, y)$ be the number of topplings at site (x, y), after adding N grains and relaxing the system completely. We define the function

$$\Phi(\xi, \eta) = \lim_{N\to\infty} \frac{1}{\Lambda^2} T_N([\xi\Lambda], [y\Lambda]), \tag{1}$$

where $[x]$ denotes the integer nearest to x. The conservation of number of grains implies that the potential function satisfies the Laplace's equation [5]

$$\nabla^2\Phi(\xi, \eta) = -\delta(\xi, \eta) + \Delta\rho(\xi, \eta). \tag{2}$$

In an electrostatic analogy, we can think of $\Phi(\xi, \eta)$ as the potential produced by a unit positive point charge at the origin, and an areal charge density $-\Delta\rho(\xi, \eta)$. We shall refer to $\Phi(\xi, \eta)$ as the potential function hereafter. In each periodic patch, the potential $\Phi(\xi, \eta)$ is a quadratic function of the coordinates ξ and η [12]. For the pattern on the

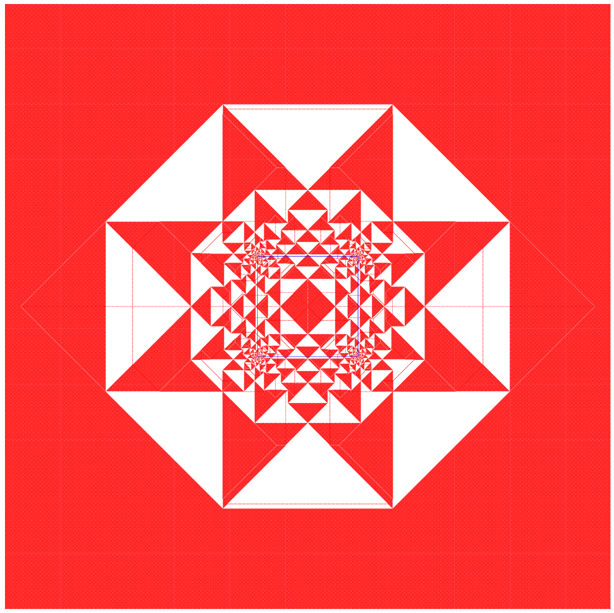

Figure 5. A pattern similar to the one in figure 4, but this time the grains are added uniformly, four at every site inside the square of width 160 lattice units. The diameter of the full pattern is 592. Color code: Red=0, White=1.

F-lattice, it was shown that the coefficients of the quadratic terms are simple rational numbers, while the linear terms can be determined by the condition that the potential and its derivative are continuous functions at the boundaries where two patches meet. This allowed us to characterize the asymptotic pattern completely [5].

3. Effect of fluctuations in the site of addition

The patterns studied so far are produced by adding one grain at each time step, at a fixed site (the origin). Now, we consider how these change when the site of addition fluctuates in time at random. To be more specific, we add N grains by randomly distributing them among sites within a small square centered at the origin. The size of the square is chosen to be of length $\epsilon\Lambda$, with $\epsilon < 1$. The background is a checkerboard distribution of heights 1 and 0.

We have shown the pattern for $\epsilon = 0.3$ and $N = 120,000$ in figure 4a, and the pattern corresponding to $\epsilon = 0.1$ and $N = 75,000$ in figure 4b. We see that the patches away from the boundary of the square region of addition, are identical to that of the single source pattern (compare with figure 2). That is, the patches at the outer layers in the pattern, are not much affected by this change. Only near the center, within a distance of order ϵ, we see a change. Near the center, the dense set of patches of decreasing size is smeared out, and there are no periodic quadratic patches left. However, there are new accumulation points of patches, which develop at the four corners of the square region of addition. As more grains are added, finer patches appear in a way that

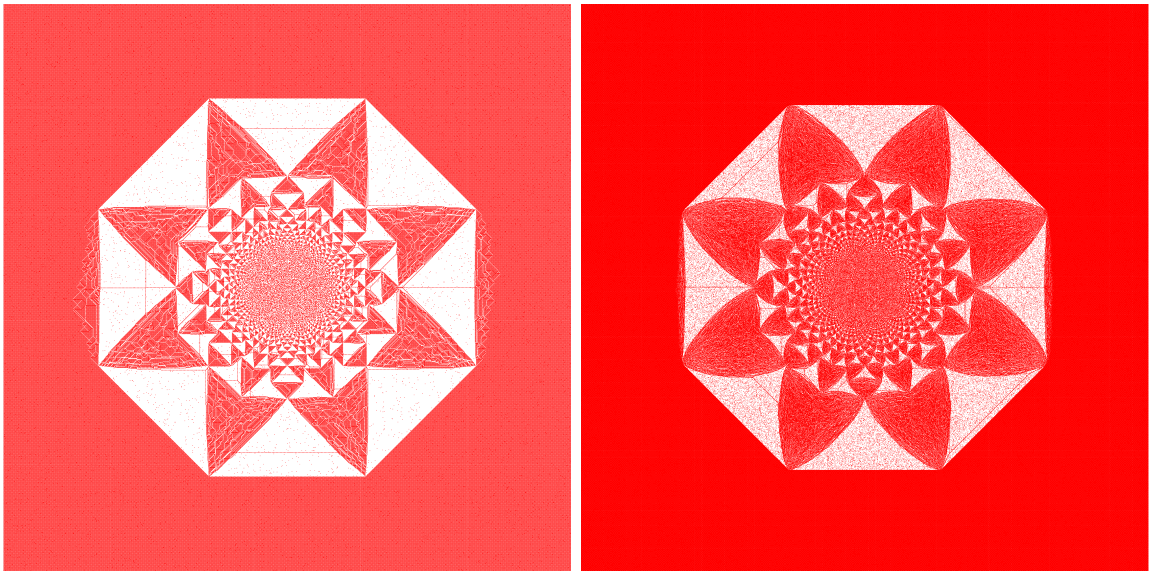

Figure 6. The patterns produced on the F-lattice by adding $N = 228,000$ and $N = 896,000$ grains at a single site on a background of mostly checkerboard distribution, except height 1 at 1% and 10% of the sites, respectively, are replaced by height zero. Color code: Red=0, White=1.

their number would become infinite in the asymptotic limit of large N. For small ϵ, the shape of the outer parts of the pattern shows only a weak dependence on its value.

The pattern changes in an interesting way, when the addition of grains are not random. The pattern produced by uniformly adding $m = 4$ grains at every site inside a square of size $2l \times 2l$ is shown in figure 5. The boundary of the square is indicated by a solid blue line. The pattern outside and away from the square, looks very similar to the pattern corresponding to random addition. The most significant change is inside the square, where there are well defined periodic patches. Again, there is no accumulation point of tinier and tinier patches at the origin. As we increase l, keeping m fixed, more and more patches appear near the corner of the square. It seems that their number would tend to infinity for large N.

4. Randomness in the initial configuration

The patterns show a significant amount of robustness to the noise in the background. The least effect of change in the background on F-lattice occurs if some heights 1 are replaced by heights zero.

This is easy to see using the abelian property of the ASM. Let C be the initial height configuration and D be the final configuration produced by adding N grains at the origin. Consider a particular site i, which has height 1 in both C and D. Let the configurations obtained from C and D by changing the height at this site from 1 to 0 be called C' and D', respectively. Then, if C' and D' contain no forbidden sub-configurations [18], using the abelian property one can show that addition of N grains

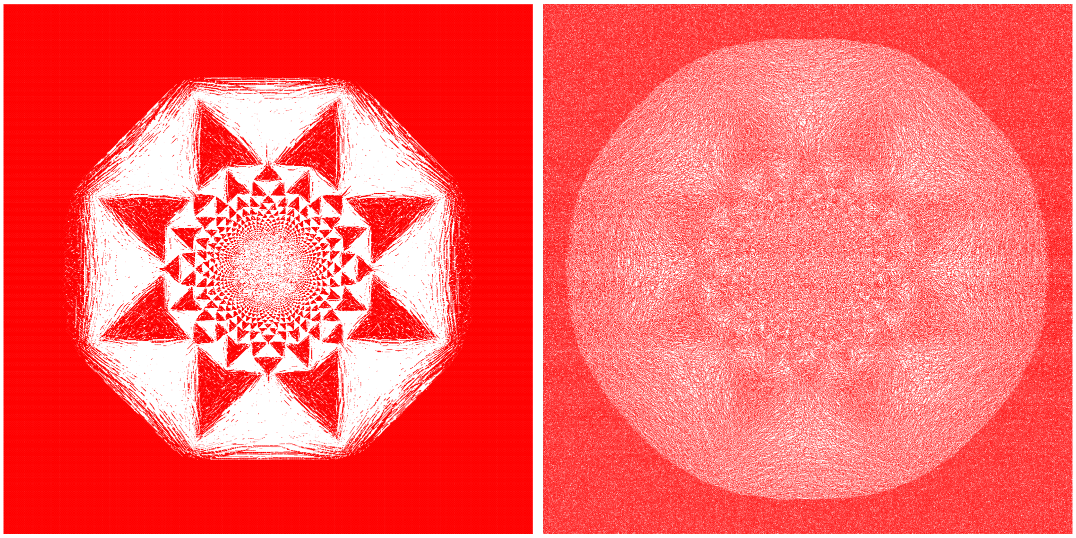

Figure 7. The patterns produced on a mostly checkerboard background, except heights at 1% and 10% sites, respectively, are flipped. Color code: Red=0, White=1.

in C' would give relaxed configuration D'. Also the toppling function is same in both the cases.

Thus we expect that a very small concentration of 1's replaced by 0's will have only a small effect. This expectation is verified numerically. The patterns on the F-lattice corresponding to backgrounds with 1% and 10% noise in the background are shown in figure 6a and figure 6b, respectively. We see that the qualitative structure and placement of different patches is unaffected in the low noise case. In particular, there are only two types of patches, and the relative positions and sizes of the larger patches are not changed much. The excess density is uniform within each patch, and jumps sharply across clearly defined patch boundaries. The outer boundary of the pattern, separating the red region outside and the eight largest white patches, seems to remain a nearly perfect octagon, but with slightly rounded off corners. However, other patch boundaries are no longer straight lines in the presence of noise, and a significant curvature in the patch boundaries is clearly seen in patterns for larger noise.

Even for the relatively large noise value (Figure 6b), the basic eight petal structure of the pattern without noise is clearly visible. However with increase of defect density, the relative area of the dense patches (white color) decreases. Also the corners of the outer boundary of the pattern becomes smoother, and for defect density close to 50%, the pattern becomes a circle with a single aperiodic patch inside.

The patterns are more sensitive to the changes in heights 0's to 1's. In figures 7a and 7b, we have shown the resulting patterns when in the initial background pattern, the heights at a fraction ϵ of randomly chosen sites are flipped from 1 to 0, and vice versa. The mean density of the background remains 1/2. The patterns correspond to $\epsilon = 0.01$ and 0.1, respectively. In this case, the most noticeable qualitative changes seems to be the fact that boundaries between patches are no longer sharp, which makes

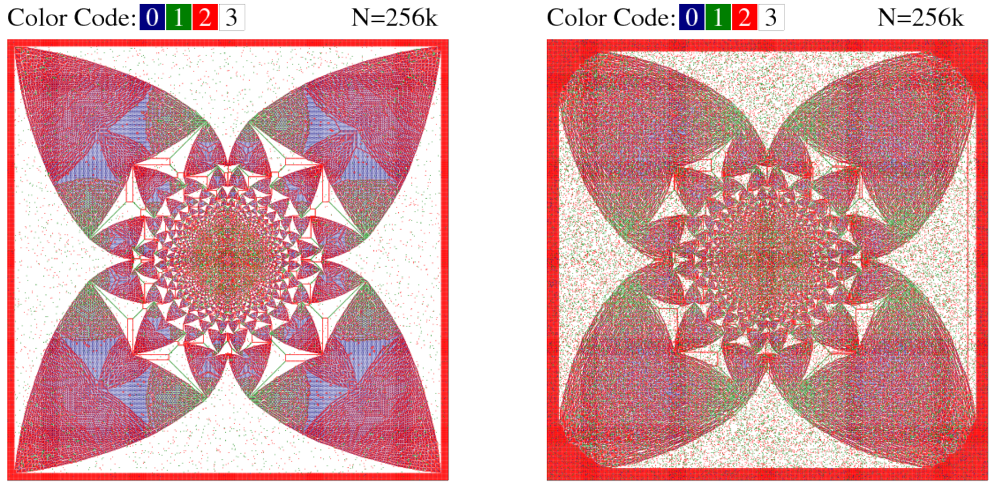

Figure 8. The patterns produced on a square lattice with a background in which heights $z = 2$ at most of the sites except, 1% and 10% of the sites, respectively, are with random assignment of heights 0 or 1. Color code: Red=0, White=1.

even a precise definition of a patch difficult.

Comparatively, the patterns in an ASM on a undirected square lattice are more robust against addition of small amount of randomness in background. We introduce randomness in the uniform background of height 2 by assigning each site heights $0, 1$ or 2 with probabilities $p/2, p/2$ and $1 - p$, respectively, independent of other sites.

The patterns corresponding to $p = 0.01$ and $p = 0.1$ are shown in figure 8a and 8b, respectively. These should be compared with the pattern produced by adding $N = 250,000$ grains on a background of height 2 at all sites, shown in figure 3.

At $p = 0.01$, the patches with height predominantly 3, do not change much, except the presence of reduced heights at the defect sites. This is easy to understand using an argument similar to the one given for the F-lattice pattern. The presence of defect sites inside the rest of the regions, generates line-discontinuities in the heights, which washes out the smaller features of the pattern. As the noise is increased, the number of defect lines increases, and the finer features are lost. Also the corners of the outer boundary become round. At noise strength $p \simeq 0.5$, the pattern becomes a circle with random distribution of heights of uniform density, inside.

We note that, other types of the randomness in the initial conditions can have different behavior. For example, den Boer *et.al.* [16] have shown that if one adds an arbitrary small density of sites with height 3, while all the other sites have height 2 on the undirected square lattice, one gets infinite avalanches for *finite* N, with probability 1.

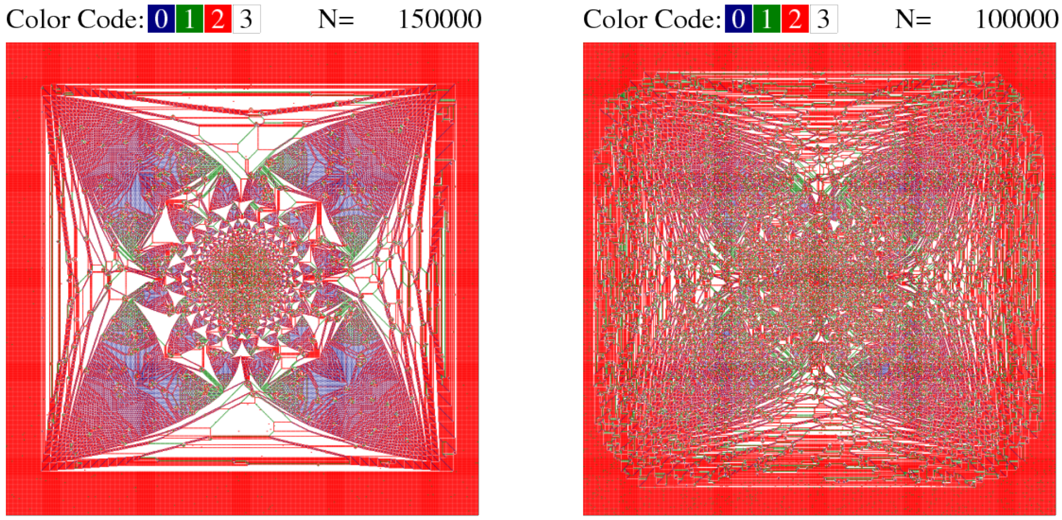

Figure 9. The patterns produced on a square lattice with 1% and 10% of the edges, respectively, are broken. Initial configuration with all heights 2.

5. Randomness in the Toppling matrix

We now consider the effect of disorder in the underlying lattice on which the growth occurs. We have considered two types of disorders. The first one is a bond disorder, where a randomly chosen fraction ϵ of the bonds are removed. Here, no grain transfer can occur along these bonds. For the undirected square lattice case, to keep the conservation law of sand in the model, we change the critical height at the end points of each such broken bond, so that a site becomes unstable when its height equals or exceeds its coordination number. The toppling rules are deterministic, and the number of sand grains are conserved in a toppling. However, the toppling matrix is no longer translationally invariant.

The patterns corresponding to the undirected ASM on the square lattice with 1% and 10% broken bonds are shown in figure 9. We see that even for $\epsilon = 0.01$, the pattern has changed substantially. The outermost patches, which, in the absence of noise, had all sites with height 3, now show a large number of criss-crossing defect lines. Further, counting inwards from outside, one can clearly see at least three or four more rings of patches. Fewer features are clearly distinguishable, for larger noise. However, some remnant of the characteristic four-petal pattern of the noise-free case, can be clearly seen even for $\epsilon = 0.10$.

The second type of disorder that we considered, is a type of site-disorder. We consider the F-lattice, where a small fraction ϵ of the sites are chosen at random, and we change the direction of bonds going out (from up-down to left-right, and vice-versa). The critical height remains unchanged, and is the same for all sites. However now, at each site the number of in-arrows is not necessarily equal to the number of out-arrows.

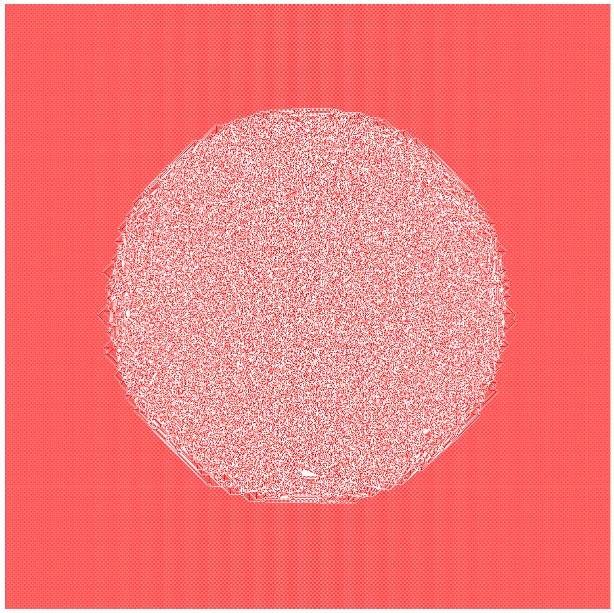

Figure 10. A pattern produced on a F-lattice in which 1% of the sites has incoming and outgoing arrows switched. Color code: Red=0, White=1.

As discussed by Karmakar *et.al.* [19], this is a relevant perturbation, and an arbitrarily small ϵ changes the critical exponents of the avalanches. We find that the patterns in growing sandpiles are also unstable to even a little amount of this kind of randomness. The pattern corresponding to $p = 0.01$ and $N = 57,000$ is shown in figure 10. This pattern is a circle with no distinguishable structures inside.

6. Effect of randomness in the toppling

We have also studied the effect of noise in the toppling rules. We have considered the F-lattice. For each toppling at a site, the direction of the outgoing grains differs from the direction of the outgoing arrows, with a probability ϵ. The two grains go in the direction of outgoing arrows with probability $1 - \epsilon$, while they go in the direction of incoming arrows with probability ϵ. The stochastic toppling rules take this modification of the ASM to the Manna universality class, which is different from that of the deterministic ASM with fixed toppling rules. In this case, we expect that the pattern would be unstable against such perturbations. This is verified by our simulations. We simulated the pattern obtained by adding $57,000$ grains on the F-lattice with checkerboard background and $\epsilon = 0.01$. The resulting pattern is a simple, nearly circular blob, with no other discernible features. It is visually indistinguishable from the pattern in figure 10.

7. Discussion

The complicated and beautiful patterns produced in the growing sandpiles are the result of an interplay between macroscopic conservation laws (encoded in the Poisson's equation satisfied by the potential function) and the integer nature of the microscopic variables. This is not yet well understood. In fact, starting from the ASM rules, as yet we can not prove even the existence of proportionate growth in the growing patterns.

In the presence of noise, an analytical study of this problem is even more difficult. Clearly, the potential function is no longer piece-wise quadratic, and the analytical techniques used in [5], to characterize the pattern exactly, no longer work. In fact, in figures 7 and 9, there are no sharp patch boundaries, and perhaps one can not even give a clear definition of 'patches', at all. The patterns are characterized by the nontrivial spatial dependence of the density function $\Delta\rho(\xi, \eta)$. The pictures are reminiscent of Rayleigh-Bénard convection patterns [20], where a linear analysis about the uniform steady state shows that, in some regime of parameters, they become unstable to a class of space-dependent perturbations. In our numerical studies, we see that the featureless circular-blob-pattern of growth at high noise levels, is unstable with respect to some characteristic low-wavelength density instabilities for low-noise deterministic models. However, there is an important difference between these two cases. In the convection problem, the amplitude of the perturbations grows in time exponentially till it reaches a saturation value, determined by the nonlinear terms, whereas in the sandpile problem, the notion of "growth of amplitude in time" is not well-defined.

Nevertheless, for the sandpile patterns there is the "least action principle", which is a variational principle that allows us to compare different trial patterns, and select the pattern corresponding to the minimum action. Here 'action' is measured in terms of the total number of toppling events. The principle, in the ASM context, is informally stated as the lazy man's maxim: "Don't do anything, unless you have to". If we think of topplings as dissipative events, it is similar in spirit to the principle of minimal heat production in resistor networks, or the minimum entropy production principle, often discussed in non-equilibrium statistical physics. While the extent of validity of the latter, in general, is not clearly established (see, for example, the discussion in [21]), for this special case of ASM's with a threshold rule for topplings, given that the order of topplings does not matter, the principle is easily proved, and is more or less built into the rules of evolution [16].

More precisely, if one considers a starting configuration C of an ASM, with one or more unstable sites, then the toppling rules of the ASM determine the stable final configuration F, uniquely. Suppose we modify the toppling rules of the ASM by dropping the condition that a toppling occurs only at sites where the height exceeds the threshold value, and allowing topplings at any site. For example, starting with a configuration of all sites with height zero on the undirected square lattice, a toppling at the origin would make the height there -4, and height at each of the neighbors, $+1$. Then, starting from C, there is a large number of stable configurations reachable by topplings. The 'least

action principle' for the ASM states that, if we can reach a *stable* configuration F' from C under this less restrictive dynamics, the number of topplings required to reach F' is greater than that required to reach F, for all $F' \neq F$.

The variational principle allows us to compare different guesses for the final configuration, and tells us which one is closer to the actual pattern. The main difficulty in applying this principle, in practice, is that the set of configurations, over which the extremization has to be done, is all possible configurations *reachable from the initial unstable pattern by topplings*. Characterizing this set is rather difficult. However, one can restate this principle in terms of non-negative integer toppling functions, $T_N(x, y)$. For any choice of $T_N(x, y)$, there is a well-defined, easily computed, final height configuration. Also, one can systematically improve on an initial trial function, by performing more topplings at sites which are unstable in the final configuration, or by untoppling at sites where the heights are too low. This has been shown to yield a very efficient algorithm for determining the final configuration for the related rotor-router model [22].

There are other questions that have been addressed only partially in this paper. Certainly, it would be useful to have a more quantitative study of the patterns using the toppling function. Numerical studies with significantly larger N, can clarify whether or not the density function shows spatial discontinuities in the presence of low noise of the type shown in figures 6 and 8, and whether the excess density is exactly a constant within a patch. It is hoped that further work will clarify these issues.

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